Math Manipulatives: Making the Abstract Tangible

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Abstract

In order to succeed in mathematics students must develop an understanding of abstract concepts. Elementary teachers often use math manipulatives to represent concretely the abstract concepts that students are learning and to connect these concepts to prior knowledge. Traditionally, teachers and students used concrete manipulatives, however, in many contemporary classrooms teachers and students also use pictorial and virtual manipulatives. This article will begin by defining, and providing examples and potential applications of concrete, pictorial, and virtual manipulatives. Next, this article will present the theoretical foundations for teachers and students to use manipulatives in mathematics education. Finally, this article will review the literature on the impacts of teachers and students using math manipulatives. The author hopes that this article reveals the factors and conditions that may contribute to educators’ successes and struggles with using manipulatives.

Keywords: mathematics, mathematics education, manipulative, representation
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Teaching for understanding in mathematics involves presenting the curriculum through multiple representations. Representations, which can be used in mathematics education, include physical (concrete), pictorial (static visual), and virtual (dynamic electronic) representations. In addition to the abstract (also referred to as symbolic) representation, math educators can use math manipulatives to model multiple representations of math concepts. Each major math standards document published in the past 20 years has advocated for the use of manipulatives. The fourth Mathematical Practice Standard in the Common Core State Standards for Mathematics (hereafter CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), “model with mathematics” highlights students’ use of models to communicate their thinking when solving math problems. Teachers striving to develop students’ capacity to "model with mathematics" should explicitly make connections between real-world scenarios and mathematical representations of those scenarios during their instruction. Moreover, The National Council of Teachers of Mathematics (NCTM, 2000) Principles and Standards for School Mathematics (PSSM) encourage teachers and students to use multiple representations during mathematics instruction. The PSSM states that all students should “create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; [and] use representations to model and interpret physical, social, and mathematical phenomena” (p. 67).

The primary purpose of this research review is to survey the literature regarding the effectiveness of using physical, pictorial, and virtual manipulatives at enhancing students’ conceptual understanding of abstract concepts. Prior to evaluating the efficacy of physical, pictorial, and virtual manipulatives, this article will define each form as well as their potential applications. Next, this article will examine the theoretical foundation supporting the use of manipulatives. Subsequently, this article will present the findings of contemporary research studies on the advantages and disadvantages of the use of manipulatives. The author hopes that this research review uncovers the factors and conditions that contribute to educators’ successes and struggles with using manipulatives.

Physical manipulatives

A physical manipulative is an object, “designed to be moved or arranged by hand as a means of developing motor skills or understanding abstractions, especially in mathematics” (“Manipulative,” 2009). Physical manipulatives range from low-cost, simple, everyday items, such as buttons, paper clips, toothpicks, dominoes, money, string, playing cards, rulers, number cubes, graph paper, empty egg cartons, measuring cups, and film canisters to more complex and discipline-specific items, such as calculators, two-color counters, thermometers, decimal tiles, pattern blocks, Cuisenaire rods, geo-boards, tangrams, algebra tiles, and pentominoes (Bellonio, 2001). According to CCSS-HSA.APR.A.1, students in high school should develop an understanding of how to “multiply polynomials” (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). As seen in Figure 1, students can use algebra tiles to model the multiplication of polynomials.
Figure 1: Using algebra tiles (a physical manipulative) to model the multiplication of polynomials

**Pictorial manipulatives**

A pictorial manipulative is a stationary model that helps students visualize math concepts. According to Muser, Peterson, and Burger (2014) drawing a picture may be helpful when the learner wants to gain a better understanding of the problem, when a visual representation of the problem is possible, or when the problem involves a physical situation, geometric figures or measurements. It is important to note that a physical manipulative can be represented as pictorial manipulative (by creating a drawing of it), but the pictorial manipulative will lack the tangible and dynamic attributes of the physical manipulative. According to CCSS-7.NS.A1, students in 7th grade should develop an understanding of how to “add and subtract rational numbers” (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). As seen in Figure 2, a pictorial representation of a number line can be an effective method for modeling the addition of integers.

**Virtual manipulatives**

Virtual manipulatives are “computer based renditions of common mathematics.”
manipulatives and tools” (Dorward, 2002, p. 329) and “an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge” (Moyer, Bolyard, & Spikell, 2002, p. 373). Virtual manipulatives can develop students’ visualization skills by connecting words, pictures, and symbols simultaneously. The key difference between a virtual and a pictorial manipulative is that virtual manipulatives are dynamic, while pictorial manipulatives are static. The main factor that distinguishes virtual and physical manipulatives is that virtual manipulatives are digital and therefore two dimensional, while physical manipulatives are three-dimensional. According to CCSS.8.G.B.7, students in 8th grade should be able to “apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions” (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

As seen in Figure 3, a geoboard virtual manipulative (National Library of Virtual Manipulatives, 2015) can be an effective method for modeling the two dimensional right triangles.

Figure 3: Using a virtual geoboard manipulative to model a right triangle

The next generation of high stakes assessments includes virtual manipulatives, therefore, it is imperative that learners understand how to utilize these tools. States that adopted the Common Core State Standards (CCSS) had the option of either collaborating with other states to create a CCSS aligned assessment or independently seeking a vendor for an assessment. The states interested in collaborating formed two organizations, the Partnership for Assessment of Readiness for College and Careers (PARCC) and the Smarter Balanced Assessment Consortium (SBAC). Sample items from both the PARCC and the SBAC math assessments include virtual manipulatives, such as drag and drop objects, an equation editor, and virtual number lines and graphs. Alarmingly, the amount of practice needed for learners to understand how to use these tools is unknown. Moreover, many schools lack the technology resources to provide all students with practice using virtual manipulatives.

A Theoretical Foundation for Manipulatives

Ancient civilizations throughout the world have used physical objects and drawn pictures to model math problems. The first formal uses of manipulatives in math education occurred in the late 1800s. Friedrich Froebel and Maria Montessori were two pioneers in using manipulatives in elementary math education. Since the early 1900s, it has been essential for
teachers and students to use manipulatives during elementary math teaching and learning. In fact some states, such as California, North Carolina, Texas, and Tennessee have mandated the use of manipulatives for teaching elementary math concepts ("Research on the Benefits of Manipulatives," 2015).

There is extensive theoretical support for the use of manipulatives. The work of numerous highly esteemed learning theorists, such as Piaget, Dienes, and Bruner, supports the use of physical manipulatives. There are two overarching ideas shared by these theorists. The first idea that these theorists shared is that using concrete tools is an important stage for learners as they develop an understanding of new concepts. Secondly, each theorist believed that learners’ benefit from interacting with their environment in authentic ways.

Piaget (1952) suggested that youngsters’ and adults’ thoughts, language, and actions differ in both quantity and quality. According to Piaget (1971), learners move through four stages of intellectual development (sensorimotor, preoperational, concrete operations, and formal operations) and learning involves either add new information to existing psychological frameworks (assimilation) or developing or evolving new cognitive structures (accommodation). Piaget hypothesized that children were not mentally mature enough to grasp abstract mathematical concepts if their teachers only presented the concepts in writing (using words, numbers and symbols). According to Piaget, children need several experiences with concrete materials and drawings in order to learn abstract concepts. Piaget believed that as children mature to their adolescence their need for concrete experiences diminishes but never ceases.

Dienes (1960) proposed that learners’ whose mathematical understandings were firmly grounded in manipulative experiences would be more likely to make connections between the world in which they live and the abstract world of mathematics. According to Diene’s Dynamic Principle (1971), learning is a cyclical process that begins with informal play; progresses into structured play; develops into an understanding of abstractions; and continues with a reapplication of these abstractions to more sophisticated play and learning. According to Diene’s Perceptual Variability Principle (1971), learners benefit from their teachers exposing them to the same concept multiple times using a variety of materials. Dienes believed that learners would develop a better understanding of the concept by making connections between these diverse experiences. According to Diene’s Mathematical Variability Principle (1971), teachers who present multiple examples of a concept to their students where they hold the relevant variables constant and systematically alter the irrelevant variables will be more likely to have students who are able to generalize about the concept. According to Diene’s Constructivity Principle (1971), teachers should always provide their students with opportunities to work with concrete forms of concepts before they expect their students to analyze the abstract forms of the concepts.

There is a close alignment between most of Bruner’s writings and those of Piaget and Dienes. However, Bruner departed from Piaget in his belief that readiness for learning depends more upon effective instruction and learning experiences than when the learner reaches a particular age. Bruner (1966) asserts, “Any subject can be taught effectively in some intellectually honest form to any child at any stage of development” (p. 33). Bruner (1966) suggested that learners understanding of a concept progresses through three levels of sophistication of thought. Bruner refers to learning through concrete experiences as “enactive,”
learning through visual mediums as “iconic,” and learning through abstract symbols as “symbolic.”

**Research Supporting Manipulatives**

Many studies published between 1970 and 1990 supported the use of concrete math manipulatives. Researchers have found that students who used manipulatives during mathematical instruction typically outperformed the students who did not use the manipulatives (Driscoll, 1983; Sowell, 1989; Suydam, 1986). These findings have been replicated in studies spanning across topics, grade levels, and even ability levels. Several contemporary studies suggest that students with learning disabilities may benefit from using manipulatives more than mainstream students do (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Cass, Cates, Smith, & Jackson, 2003; Maccini & Hughes, 2000; Marsh & Cook, 1996; Witzel, Mercer, & Miller, 2003).

Regarding pictorial manipulatives, evidence suggests that they too are an important way for children to express their thinking as they develop an understanding of new math concepts (Woleck, 2001). According to Thompson & Thompson (1990), teaching children how to utilize pictures to solve problems is a highly effective problem solving strategy. According to Moreno & Mayer (1999) and VanGarderen (2006), a distinguishing competency between high and low achieving math students is the ability to visualize math concepts. Moreover, Arcavi (2003) reports that students who create, interpret, utilize, and reflect on pictorial representations may be able to link these representations to abstract concepts and subsequently develop a greater understanding of the topics. Research on multiple intelligences and students’ different learning styles supports the use of pictorial manipulatives (Gardner, 1997, 2002; Marzano, 2010).

With respect to virtual manipulatives, these tools have the potential to develop students’ visualization skills by connecting words, pictures, and symbols simultaneously (Paivio, 2007). According to Paivio (2007), this coinciding presentation may help students develop a solid understanding of math concepts. According to a recent meta-analysis, using virtual manipulatives results in a moderate effect size of 0.44 compared to other instructional treatments (Moyer-Packenham, Westenskow, & Salkind, 2012).

**Struggles Encountered Using Manipulatives**

However, there are also studies that do not support the use of manipulatives. These studies focus on students not making connections between multiple representations, learning procedures rather than understanding concepts, inability to transfer knowledge to new applications, and learners having fun with the manipulative and essentially utilizing it as a toy.

Ball points out that manipulatives are not magical transmitters of meaning or insight (Ball, 1992). According to Ball (1992), “although kinesthetic experience can enhance perception and thinking, understanding does not travel through the fingertips and up the arm” (p. 47). Boulton-Lewis (1998) found that conceptual processing could not occur unless the student has reached a point of automaticity with the manipulative. A manipulative is an artifact not a tool if the student is constantly aware of it. In this situation, the manipulative is not helping the student.
move toward a goal state of understanding (Winograd and Flores, 1986).

Learners often use manipulatives in a mechanical manner, with little or no learning of the mathematical concepts behind the procedures (Hiebert and Wearne, 1992). As a result, students are often unable to make connections between their actions with manipulatives and abstract symbols (Thompson & Thompson, 1990).

Meira (1998) defines the concept of transparency of instructional devices as “an index of access to knowledge and activities rather than as an inherent feature of objects . . . a process mediated by unfolding activities and users’ participation in ongoing sociocultural practices” (p.121). Learners must reach transparency in order to develop a conceptual understanding. Learners can reach transparency with one application of a manipulative, but fail to reach it with a different application of the same manipulative. Teachers can assess transparency through observation and inquiry. A student who has reached transparency will be able to go through the appropriate motions without assistance. They will also have the ability to explain what they are doing, why they are doing it, and how their procedures apply to the given concept. Learners develop a conceptual understanding through repeated practice and reflection.

**Recommendations for Manipulative Use**

While studies have demonstrated that manipulatives have the potential to help make abstract ideas concrete, provide 3D models, create a vehicle for testing and confirming reasoning, aid in problem solving, make learning math interesting and enjoyable, and build student confidence, Baroody (1989) points out that the positive results depend upon the conditions under which the manipulatives are being used. Teachers who follow the guidelines in Table 1 are more likely to experience positive results (Dunlap & Brennan, 1979, p. 90; Ross & Kutz, 1993, p. 256).

**Table 1**

*Guidelines for Manipulative Use*

- The manipulatives needs to reinforce the objectives of the lesson.
- The manipulatives must correctly portray the actual math process or concept.
- Learners should have the opportunity to use a variety of manipulatives when their teachers first introduce a new math process or concept.
- The manipulative should have moving parts that learners use to illustrate the math process concept.
- Learners need explicit instruction on how to use manipulatives.
- There should be a phasing out of the learners use of the manipulatives as they develop and understanding of the math process or concept.
- The learners must interact with the manipulatives in order to develop an understanding of the math process or concept.
- The teacher should give each learner the opportunity for individual exploration with the manipulatives.
- The math process or concept portrayed by the manipulative must be associated with the pencil and paper representation of the math process or concept.
- The learners will need explicit instruction on using their metacognitive skills.

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Conclusion

Through the review of the research on the effectiveness of using math manipulatives, the author uncovered numerous studies that supported the use of concrete, pictorial, and virtual manipulatives. For decades, researchers have been demonstrating the positive effects of using concrete and pictorial math manipulatives with their students. Studies that are more contemporary have extended these findings to virtual manipulatives. Although some studies have reported insignificant or even negative results when using math manipulatives, it seems that these outcomes are associated with the instructional methods used. Educators who follow the best practice recommendations for manipulative use are likely to experience positive outcomes.

As time passes more and more educational materials (i.e. textbooks, homework assignments, and tests) are available virtually in a digital format. Therefore, it is critical that teachers receive the necessary pedagogical training on how to use these materials. Moreover, teachers should not assume that students would automatically understand how to use these materials independently. Rather, explicit instruction and scaffolding of supports is necessary. Future research on the best practices for training teachers and students would be helpful. It is clear that waiting until a few days before an online high-stakes test to learn how to use the virtual manipulatives included in the test is a recipe for failure. However, exactly how much instruction teachers and students need as well as the exact nature of this instruction is yet to be determined.

References


